Math 485
Fall 2013
Midterm 2 - Form A
11/14/2013

Name (Print): ______________________

This exam contains 6 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note (1 sided) and a scientific calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. For example, in question involved the multi-period binomial model, I would like to see how you derive the no arbitrage price, say by displaying the tree with all the nodes filled out if the situation is appropriate.
- If the answer involves the probability of a well-known distribution, says the Normal(0,1), **you can leave the answer in the form** $P(Z > x)$ **or** $P(Z < x)$ where x is a number you found from the problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	15	
4	10	
5	15	
6	10	
Total:	100	

The following information may be useful for the exam.
1. If X has distribution $N(0, \sigma^2)$ then

$$E(e^X) = e^{\frac{\sigma^2}{2}}.$$

2. If Δ_t is deterministic, then $\int_0^t \Delta_s dB_s$ has distribution $N(0, \int_0^t \Delta_s^2 d_s)$.

3. For all random processes Δ_t given in this exam:

$$E(\int_0^t \Delta_s d_s) = \int_0^t E(\Delta_s) ds.$$

4. For all random processes Δ_t given in this exam:

$$E(\int_0^t \Delta_s dB_s) = 0.$$

5. If X has distribution $N(0, \sigma^2)$ then $E(X^4) = 3\sigma^4$.

6. 6. If X has distribution $N(\mu, \sigma^2)$ then X has density

$$\phi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\big(-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\big).$$

7. Partial credits will be given so it is to your advantage to attempt all the questions and not leave any problems blank.

1. (25 points) Let S_t represent the stock price under the risk neutral measure using the Geometric Brownian motion model,i.e.

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Derive the Black-Scholes formula for the price V_0 of the European Call with strike price K, expiration time T, i.e. $V_T = (S_T - K)^+$.

Ans: See class notes.

2. (25 points) Let S_t represent the stock price under the risk neutral measure using the Geometric Brownian motion model,i.e.

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Let $V(t, S_t)$ be the price of a Euro-style derivative that pays $\phi(S_T)$ at time T. Derive the partial differential equation that $V(t, x)$ has to satisfy.

Ans: See class notes.

3. Let $0 < s < t < T$. Compute the following

(a) (5 points) $E(B_t^2 B_s^2)$.
Ans:

$$\begin{aligned}
E(B_t^2 B_s^2) &= E((B_s + B_t - B_s)^2 B_s^2) \\
&= E\big[(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2)B_s^2\big] \\
&= E\big[B_s^4 + 2B_s^3(B_t - B_s) + B_s^2(B_t - B_s)^2\big] \\
&= E(B_s^4) + 2E(B_s^3)E(B_t - B_s) + E(B_s^2)E((B_t - B_s)^2) \\
&= 3s^2 + s(t - s) = 2s^2 + st.
\end{aligned}$$

(b) (5 points) $E(e^{B_s^2 + B_t - B_s})$.
Ans:

$$\begin{aligned}
E(e^{B_s^2 + B_t - B_s}) &= E(e^{B_s^2} e^{B_t - B_s}) \\
&= E(e^{B_s^2})E(e^{B_t - B_s}) \\
&= \frac{1}{\sqrt{1 - 2s}} e^{\frac{1}{2}(t-s)}.
\end{aligned}$$

(For the calculation of $E(e^{B_s^2})$ see part c.

(c) (5 points) $E(e^{B_t^2})$.

Ans:

$$
\begin{aligned}
E(e^{B_t^2}) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{x^2} e^{-\frac{x^2}{2t}}\, dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\frac{(2t-1)x^2}{2t}}\, dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\frac{t}{1-2t}}}\, dx \\
&= \frac{1}{\sqrt{1-2t}} \frac{1}{\sqrt{2\pi\frac{t}{1-2t}}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\frac{t}{1-2t}}}\, dx \\
&= \frac{1}{\sqrt{1-2t}},
\end{aligned}
$$

since $\frac{1}{\sqrt{2\pi\frac{t}{1-2t}}} e^{-\frac{x^2}{2\frac{t}{1-2t}}}$ is the density of $N(0, \frac{t}{1-2t})$. Note that we also require $t \leq \frac{1}{2}$ here for the calculation to make sense.

4. (a) (5 points) Compute $d(\sin(t)B_t^2)$.
   Ans:

$$
\begin{aligned}
d(\sin(t)B_t^2) &= \cos(t)B_t^2 dt + 2\sin(t)B_t dB_t + \sin(t)dt \\
&= (\sin(t) + \cos(t)B_t^2)dt + 2\sin(t)B_t dB_t.
\end{aligned}
$$

   (b) (5 points) Compute $E(B_t^6)$. (Hint: Apply Ito formula to compute dB_t^6.)

$$dB_t^6 = 6B_t^5 dB_t + 15B_t^4 dt.$$

Hence

$$B_t^6 = \int_0^t 6B_s^5 dB_s + \int_0^t 15B_s^4 ds.$$

Hence

$$
\begin{aligned}
E(B_t^6) &= \int_0^t 15E(B_s^4) ds \\
&= \int_0^t 45s^2 ds = 15t^3.
\end{aligned}
$$

5. Let S_t represent the stock price under the risk neutral measure using the Geometric Brownian motion model,i.e.

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

   (a) (7 points) Compute the price $V(t, S_t)$ at time t of a Euro-stlye derivative that pays $V_T = (S_T - K)^2$ at time T.

$$\begin{aligned} V(t,S_t) &= E\big(e^{-r(T-t)}(S_T-K)^2|S_t\big) \\ &= E\big(e^{-r(T-t)}S_T^2 - 2Ke^{-r(T-t)}S_T + e^{-r(T-t)}K^2|S_t\big) \end{aligned}$$

$$S_T^2 = S_t^2 \exp\big(2(r-\frac{1}{2}\sigma^2)(T-t) + 2\sigma(B_T-B_t)\big)$$

Hence

$$E\big(e^{-r(T-t)}S_T^2|S_t\big) = S_t^2 e^{(r-\sigma^2)(T-t)}e^{2\sigma^2(T-t)} = S_t^2 e^{(r+\sigma^2)(T-t)}.$$

So

$$V(t,S_t) = S_t^2 e^{(r+\sigma^2)(T-t)} - 2KS_t + e^{-r(T-t)}K^2.$$

(b) (8 points) Verify that the price $V(t,x)$ you derived in part a satisfies the Black-Scholes PDE. We have $V(t,x) = x^2 e^{(r+\sigma^2)(T-t)} - 2Kx + e^{-r(T-t)}K^2$. So

$$\begin{aligned} \frac{\partial V}{\partial t} &= -(r+\sigma^2)x^2 e^{(r+\sigma^2)(T-t)} + re^{-r(T-t)}K^2 \\ \frac{\partial V}{\partial x} &= 2xe^{(r+\sigma^2)(T-t)} - 2K \\ \frac{\partial^2 V}{\partial x^2} &= 2e^{(r+\sigma^2)(T-t)}. \end{aligned}$$

Plug into the PDE:

$$\begin{aligned} \frac{\partial V}{\partial t} &= -(r+\sigma^2)x^2 e^{(r+\sigma^2)(T-t)} + re^{-r(T-t)}K^2 \\ \frac{\partial V}{\partial x}rx &= 2rx^2 e^{(r+\sigma^2)(T-t)} - 2Krx \\ \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\sigma^2 x^2 &= e^{(r+\sigma^2)(T-t)}\sigma^2 x^2 \\ -rV &= -rx^2 e^{(r+\sigma^2)(T-t)} + 2Krx - re^{-r(T-t)}K^2. \end{aligned}$$

We can verify that the terms indeed cancel.

6. Let S_t represent the stock price under the risk neutral measure using the Geometric Brownian motion model,i.e.

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

(a) (5 points) Consider the price V_0 of a European call option on S with strike price K, expiration time T. What do you think V_0 is when K is very small compared with S_0? Justify your answer using the Black-Scholes formula. (Hint: What happens when $\frac{K}{S_0} \to 0$? Also recall $N(\infty) = 1$ and $N(-\infty) = 0$).

Ans: Since

$$d_1 = \frac{(r + \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}}$$
$$d_2 = \frac{(r - \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}},$$

we see that $d_1 \to \infty$ and $d_2 \to \infty$ as $\frac{K}{S_0} \to 0$. So

$$\begin{aligned} V_0 &= S_0 N(d_1) - e^{-rT} K N(d_2) \\ &= S_0(N(d_1) - e^{-rT}\frac{K}{S_0} N(d_2)) \\ &\to S_0 \text{ as } \frac{K}{S_0} \to 0. \end{aligned}$$

(b) (5 points) Consider the price V_0 of a European call option on S with strike price K, expiration time T. What do you think is V_0 when the volatility σ is very large? Justify your answer using the Black-Scholes formula. (Hint: What happens when $\sigma \to \infty$? Also recall $N(\infty) = 1$ and $N(-\infty) = 0$).

Ans: Since

$$d_1 = \frac{(r + \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}}$$
$$d_2 = \frac{(r - \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}},$$

we see that $d_1 \to \infty$ and $d_2 \to -\infty$ as $\sigma \to \infty$. So

$$\begin{aligned} V_0 &= S_0 N(d_1) - e^{-rT} K N(d_2) \\ &\to S_0 \text{ as } \sigma \to \infty. \end{aligned}$$